**Ch. 2 Non-paraxial/exact matrices, meridional rays, and spherical aberration**

The non-paraxial/exact matrices

**Fig. 1 Picture of refraction at surface**

From Snell’s law we have,

|  |  |  |
| --- | --- | --- |
|  |  | (1) |

From the projection of triangle PUV onto the X axis we have,

|  |  |  |
| --- | --- | --- |
|  |  | (2) |

From Fig. 1, it can also be seen that

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

Combining (1) – (3) we obtain,

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

where, is called the *skew power* of surface 1. For small angles, we have, , which essentially reduces to the paraxial approximation. Therefore, the exact matrix equation for refraction at surface 1 is,

|  |  |  |
| --- | --- | --- |
|  |  | (5) |

where, the refraction matrix is now,

|  |  |  |
| --- | --- | --- |
|  |  | (6) |

**Fig. 2 Translation between surfaces**

From Fig. 2 we can see that,

|  |  |  |
| --- | --- | --- |
|  |  |  |

Therefore, the matrix equation for translation between the two surfaces is,

|  |  |  |
| --- | --- | --- |
|  |  | (7) |

where, the translation matrix is now

|  |  |  |
| --- | --- | --- |
|  |  | (8) |

For the exact matrices describing optical systems we now must determine the actual displacement of the ray between the two surfaces and we must also find out the angles with respect to the normal of the surfaces and , as the refractive matrix now depends on the orientation of entering/exiting light rays. However, these quantities can be evaluated quite easily for meridional rays by simply knowing the object height, position relative the lens surface, inclination angle of ray and radius of curvature of the surface. To see this consider the following diagram depicting a generalised geometrical situation:

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**Fig. 3 Geometry for determination of ray displacement, , and skew power .**

and are perpendicular to the ray, and parallel to it. is the intersection of the ray with a convex surface and with a concave. Therefore,

|  |  |  |
| --- | --- | --- |
|  |  |  |

With the + corresponding to the concave surface and the – with the convex.

But, , , and .

And, .

|  |  |  |
| --- | --- | --- |
|  |  | (9) |

Thus the displacement or translation to or between surfaces can be found knowing the position of the object (, height of the object and the angle of inclination of the light ray ().

Likewise the skew power can also be found knowing only these values by,

|  |  |  |
| --- | --- | --- |
|  | And using Snell’s law () we find, | (10) |

where from Fig. 3,

|  |  |  |
| --- | --- | --- |
|  |  | (11) |

From this knowledge we can now trace exactly a ray lying in the *meriodional* or *tangential* plane, which contains the symmetry axis OZ in Fig. 1.

Example: For single lens

**Fig. 4 General lens with all variables shown in the figure using the positive convention. If the variable is in fact negative all derived formulae will still be valid.**

For a double convex lens with , , , we want to trace a ray leaving an object specified by , , from the axis and intersecting the axis on the image side.

Exact procedure:

1. First translation and refraction:

Finally, we find that

Programming all these equations we find,

radians

units

1. Second translation and refraction

We essentially repeat all proceedings steps but shift the new origin to in Fig. 4.

From, Fig. 4 we find,

Repeating the steps we can find that,

radians

1. Finding ray location where it crosses axis.

Spherical aberrations

Consider a double convex lens such as the one in Fig. 5. A ray parallel and through the central axis passes through un-deviated. If the ray is displaced from the axis but still parallel by a small amount , the paraxial approximation still holds and the ray passes through what we call the paraxial focal plane. However, as we get further from the axis though the angle with respect to the axis still remains zero, the incident and refracted angles, and increase, and therefore the paraxial approximation no longer applies and the rays exiting the lens no longer pass through the paraxial focus, leading to spherical aberration. The paraxial approximation assumes that , leading to the approximated Snell’s law . However, as get larger we must take into account the higher order terms in the Taylor series expansion:

The presence of these higher order terms means that there will be deviations from the predictions of paraxial optics, leading to defects in the image. The second term is called the *third order* or *Siedel aberrations* and the next term corresponds to the *fifth order aberrations.*

Rays parallel and infinitesimally close to the central axis, converge on the otherside through the paraxial focal plane (FP), as depicted in Fig. 6. As we get further from the central axis, the rays converge closer to the lens in comparison to paraxial focal plane. The deviations from the paraxial approximation are defined by two quantities: *a longitudunal spherical aberration (ALSA)* and a *transverse spherical aberration (TSA)* as shown in Fig. 5. Lens designers usually specify ALSA, however TSA is a natural definition which defines the focus of an image on a flat screen.

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**Fig. 5**

The rays through a ring at the edge of the lens converge at what we call the marginal focus (FM), which is the closest convergence point with respect to the lens as shown in Fig. 6. All other ensemble of parallel rays converge somewhere in-between the paraxial focal plane and the marginal focus as shown in Fig. 6. The outermost intersection of all rays which converge toward the central axis, form a surface called the caustic surface.The image rather than being a point is a circle determined by the intersection of the caustic surface with a plane. The minimum radius or the *circle of least confusion* occurs when the marginal ray crosses the central axis and intersects the caustic surface. Hence, the transverse spherical abberation may be minimised by proper focusing of the lens.

**Fig. 6**

Example: symmetrical convex lens

Consider a lens where, , , and .

1. To find approximately the paraxial focal plane using the exact matrices for the meridional plane, we essentially determine the point of convergence for a parralel incident ray close to the central axis. This is done in Python code by tracing a parallel ray originating 1200 units left of the lens () and 0.00001 units above the axis ( to the point it converges and intersects the axis.
2. Then to determine quantitatively the abberations we iteratively trace a ray further away from the axis 0.1 units at a time (. Essentially, we use equations (5)-(11) as for the previous example, but this time for each iteration. We determine the distance from lens the ray crosses the axis, for each iteration, using:
3. For each iteration we quantify the abberations using:

The result for this is,

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| 0.00001 | 9.6552 | 0 | 0 |
| 0.10001 | 9.6537 | -0.00001 | -0.0015 |
| 0.20001 | 9.6493 | -0.0001 | -0.0058 |
| ……………. | ……………. | ……………. | ……………. |
| ……………. | ……………. | ……………. | ……………. |
| ……………. | ……………. | ……………. | ……………. |
| 3.90001 | 7.7930 | -1.4535 | -2.8622 |
| 4.00001 | 6.5881 | -1.6368 | -3.0671 |

From Fig. 5, clearly the transverse spherical aberration is directly proportional to, and the longitudinal spherical aberration is directly proportional to, . Since, both these functions have the following Taylor series expansions:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

and, the deviation from the paraxial approximations are dominated by the two higher order terms in the power series, the metrics defining the aberrations can also be predicted to fitted well by the following polynomials,

|  |  |  |
| --- | --- | --- |
|  |  | (12) |
|  |  | (13) |

[Can fit with example as an exercise]

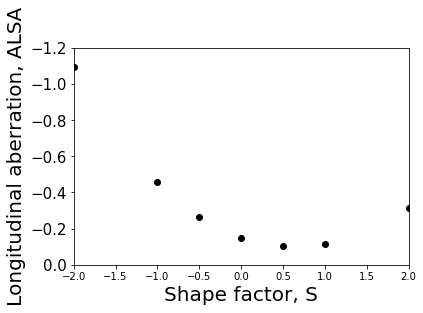
To see what we can do to minimise the spherical aberration based on the shape of the lens we shall first define the shape factor of the lens, , as:

|  |  |  |
| --- | --- | --- |
|  |  | (14) |

For symmetric lenses we have and therefore . Hence, measures the deviation of a lense from symmetry. To consider the effects of shape factor on spherical aberration we repeat effectively the previous code but for the following lenses:

|  |  |  |
| --- | --- | --- |
|  |  |  |
| -10 | -3.33 | -2 |
|  | -5 | -1 |
| 20 | -6.67 | -0.5 |
| 10 | -10 | 0 |
| 6.67 | -20 | 0.5 |
| 5 |  | 1 |
| 3.33 | 10 | 2 |

All lenses above have the same thickness, index and approximately the same focal length. Below is a plot of the longitudinal spherical aberration where object height is 1 unit (), as a function of the shape factor for the above lenses:



The minimum occurs for approximately . The process of minimising aberration based on the shape of a lens is called *bending the lens.*

For negative lenses (e.g. double concave lenses) with focal planes in the same region as the object, longitudinal sperical aberrations have an opposite sign to positive lenses with focal planes in front of the lens. To reduce aberrations we therefore usually combine both positive and negative lenses. For instance, the use of doublet type systems reduces aberrations of this sort.

We know that logitudinal aberrations take the form of:

where, is the object height.

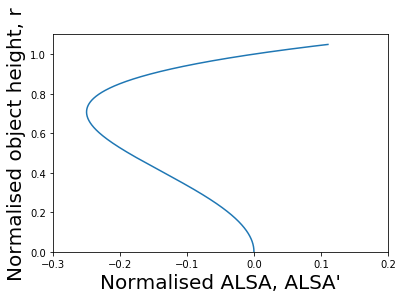
If we specify that we reduce ALSA to zero for a maximum aperture say, then,

Suggests that local maximum in longitudal aberration occurs at:

Therefore by specifying zero ALSA at we are actually overcorrecting for larger apertures. We can represent in normalised form, by using (normalised object height)

And by setting, (normalised longitudinal aberration) we have

We can now plot this as below,



Typically lens designers plot the longitudinal aberrations as a function of the object height as above.